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Gluon field strength correlation functions within a constrained instanton model

A.E. Dorokhov[1,2], S.V. Esaibegyan[1,3], A.E. Maximov[1], S.V. Mikhailov[1]

[1] Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, 141980, Moscow Region, Dubna, Russia
[2] Instituto de Fisica Teorica, UNESP, Rua Pamplona, 145, 01405-900, Sao Paulo, Brazil
[3] Yerevan Physics Institute, 375036, Yerevan, Armenia

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Abstract. We suggest a constrained instanton (CI) solution in the physical QCD vacuum which is described by large-scale vacuum field fluctuations. This solution decays exponentially at large distances. It is stable only if the interaction of the instanton with the background vacuum field is small and additional constraints are introduced. The CI solution is explicitly constructed in the ansatz form, and the two-point vacuum correlator of the gluon field strengths is calculated in the framework of the effective instanton vacuum model. At small distances the results are qualitatively similar to the single instanton case; in particular, the D_1 invariant structure is small, which is in agreement with the lattice calculations.

1 Introduction

The non-perturbative vacuum of QCD is densely populated by long-wave fluctuations of gluon and quark fields. The order parameters of this complicated state are characterized by the vacuum matrix elements of various singlet combinations of quark and gluon fields and condensates: $\langle : \bar{q}q : \rangle$, $\langle : F^a_{\mu\nu} F^a_{\mu\nu} : \rangle$,$\langle : \bar{q}(\sigma_{\mu\nu} F^a_{\mu\nu}(\lambda^a/2))q : \rangle$, etc. The non-zero quark condensate $\langle : \bar{q}q : \rangle$ is responsible for the spontaneous breakdown of chiral symmetry, and its value was estimated a long time ago within the current algebra approach. The non-zero gluon condensate $\left\langle : F^a_{\mu\nu} F^a_{\mu\nu} : \right\rangle$ through the trace anomaly provides the mass scale for the hadrons, and its value was estimated within the QCD sum rule (SR) approach. The importance of the QCD vacuum properties for hadron phenomenology has been established by Shifman, Vainshtein and Zakharov [1]. They used the operator product expansion to relate the behavior of the hadron current correlation functions at short distances to a small set of condensates. The values of low-dimensional condensates were obtained phenomenologically from the QCD SR analysis of various hadron channels.

Later the non-local vacuum condensates or vacuum correlators have been introduced [2,3]. They describe the distribution of quarks and gluons in the non-perturbative vacuum. Physically, this means that vacuum quarks and gluons can flow through the vacuum with non-zero momentum. From this point of view the standard vacuum expectation values (VEVs) like $\langle : \bar{q}q : \rangle$, $\langle : \bar{q}D^2 q : \rangle$, $\langle : g^2 F^2 : \rangle$, $\ldots$ appear as expansion coefficients of the quark $M(x) = \left\langle : \bar{q}(0)\hat{E}(0,x)q(x) : \right\rangle$ and the gluon $D^{\mu\nu,\rho\sigma}(x)$ correlators in a Taylor series in the variable $x^2/4$.

The correlator $D^{\mu\nu,\rho\sigma}(x)$ of the gluonic strengths[1],

$$D^{\mu\nu,\rho\sigma}(x-y) \equiv \left\langle : \mathrm{Tr} F^{\mu\nu}(x)\hat{E}(x,y)F^{\rho\sigma}(y)\hat{E}(y,x) : \right\rangle, \tag{1}$$

may be parameterized in the form that is consistent with the general requirements of the gauge and Lorentz symmetries as [5]:

$$\begin{aligned} D^{\mu\nu,\rho\sigma}(x) &\equiv \frac{1}{24}\left\langle : F^2 : \right\rangle \Big\{ (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})[D(x^2) + D_1(x^2)] \\ &\quad + (x_\mu x_\rho \delta_{\nu\sigma} - x_\mu x_\sigma \delta_{\nu\rho} + x_\nu x_\sigma \delta_{\mu\rho} - x_\nu x_\rho \delta_{\mu\sigma}) \\ &\quad \times \frac{\partial D_1(x^2)}{\partial x^2} \Big\}, \end{aligned} \tag{2}$$

where $\hat{E}(x,y) = P\exp\left(\mathrm{i}\int_x^y A_\mu(z)\mathrm{d}z^\mu\right)$ is the path-ordered Schwinger phase factor (the integration is performed along the *straight* line) required for gauge invariance and $A_\mu(z) = A^a_\mu(z)\lambda^a/2$, $F_{\mu\nu}(x) = F^a_{\mu\nu}(x)\lambda^a/2$, $F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + f^{abc}A^b_\mu(x)A^c_\nu(x)$. The P exponential ensures the parallel transport of color from one point to other. In (2), $\left\langle : F^2 : \right\rangle = \left\langle : F^a_{\mu\nu}(0)F^a_{\mu\nu}(0) : \right\rangle$ is a gluon condensate, and $D(x^2)$ and $D_1(x^2)$ are invariant functions which characterize non-local properties of the condensate in different directions. The form factors are normalized at zero by the conditions $D(0) = \kappa$, $D_1(0) = 1 - \kappa$, that depend on the dynamics considered. For example, for the self-dual fields $\kappa = 1$, while in the Abelian theory without monopoles the Bianchi identity yields $\kappa = 0$.

[1] We follow the convention that the coupling constant is absorbed in the gauge field $A_\nu(x)$.

The gluon correlators $D_{\mu\nu\rho\sigma}(x)$ are involved in an analysis [4] of the spectrum of the bound states of the heavy $Q\overline{Q}$ systems. The level shift depends on the parameter $\lambda\tau$, where $\tau = 4/m_Q\alpha_s^2$ is the typical time of the low lying levels of the system, and λ is the correlation length of the gluon correlator λ defined as $D(x \to \infty) \sim \langle : F^2 :\rangle \times \exp(-|x|/\lambda)$. Thus, at large distances the physically motivated asymptotics of the correlator is exponentially decreasing. The gluon correlators are the base elements of the stochastic model of vacuum [5] and in the description of high-energy hadron scattering [6].

Measuring the correlation length and vacuum field correlators was the motivation to investigate these quantities on the lattice. New high-statistical LQCD measurements of the gauge-invariant bilocal correlator of the gluon field strengths have become available down to a distance of 0.1 fm [7]. Recently, the field strength correlators have also been studied in the effective dual Abelian Higgs model in [8] and QCD sum rule approach [9]. In all these approaches (see also [10]), the exponential decay of the correlators at large distances has been observed. However, these investigations still omit a small distance behavior of the non-local condensates.

On the other hand, in QCD there is an instanton [11], a well-known non-trivial non-local vacuum solution of the classical Euclidean Yang–Mills equations with finite action and size ρ. The importance of instantons for QCD is that it is believed that an interacting instanton ensemble provides a realistic microscopic picture of the QCD vacuum in the form of an instanton liquid [12,13] (see, e.g., the recent review in [14]). It has been argued on phenomenological grounds that the distribution of instantons over their sizes is peaked at a typical value $\rho_c \approx 1.7\,\mathrm{GeV}^{-1}$ and the liquid is dilute in the sense that the mean separation between the instantons is much larger than the average instanton size.

In our previous work [15], we have shown that the instanton model of the QCD vacuum provides a way to construct non-local vacuum condensates. Within the effective single instanton (SI) approximation we have obtained the expressions for the non-local gluon $D_I^{\mu\nu,\rho\sigma}(x)$ and quark $M_I(x)$ condensates and we derived the average virtualities of the quarks λ_q^2 and gluons $\lambda_{\rm g}^2$ in the QCD vacuum. It has been found that due to the specific properties of the SI approximation (self-duality of the field) the D_1 gluon form factor is exactly zero. The behavior of the correlation functions demonstrates that in the SI approximation the model of non-local condensates can well reproduce the behavior of the quark and gluon correlators at *short distances*. Actually, the quark and gluon average virtualities, defined via the first derivatives of the non-local condensates $M_I(x^2)$, $D_I(x)$ at the origin,

$$\lambda_q^2 \equiv -\frac{8}{M_I(0)}\frac{\mathrm{d}M_I(x^2)}{\mathrm{d}x^2}\Big|_{x=0} = 2\frac{1}{\rho_c^2}\ ,$$
$$\lambda_{\rm g}^2 \equiv -8\frac{\mathrm{d}D_I(x^2)}{\mathrm{d}x^2}\Big|_{x=0} = \frac{24}{5}\frac{1}{\rho_c^2}\ , \qquad (3)$$

are connected with the VEVs that parameterize the QCD SR,

$$\lambda_q^2 \equiv \frac{\langle : \bar{q}D^2q :\rangle}{\langle : \bar{q}q :\rangle},$$
$$\lambda_{\rm g}^2 \equiv \frac{\left\langle : F^a_{\mu\nu}\tilde{D}^2F^a_{\mu\nu} :\right\rangle}{\langle : F^2 :\rangle} = 2\frac{\langle : fF^3 :\rangle}{\langle : F^2 :\rangle} - 2\frac{\langle : g^4J^2 :\rangle}{\langle : F^2 :\rangle}, \quad (4)$$

where $\langle : fF^3 :\rangle = \langle : f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu} :\rangle$, $J^2 = J^a_\mu J^a_\mu$ and $J^a_\mu = \bar{q}(x)(\lambda^a/2)\gamma_\mu q(x)$. The value of $\lambda_q^2 \approx 0.5\,\mathrm{GeV}^2$ estimated in the QCD SR analysis [16,17] is reproduced at $\rho_c \approx 2\,\mathrm{GeV}^{-1}$. This number is close to the estimate from the phenomenology of the QCD vacuum in the instanton liquid model. The model provides parameterless predictions for the ratio $\lambda_{\rm g}^2/\lambda_q^2 = 12/5$. In [15] the effect of the inclusion of the Schwinger exponent into the semi-classical calculations has been analyzed. For some quantities this effect is very strong, being of an order of 100% for the gluon and quark average virtualities.

Nevertheless, the SI approximation used evidently fails in the description of physically motivated distributions at large distances. Asymptotically, we have found $M_I(x \to \infty) \sim \rho_c^2/x^2$ and $D_I(x \to \infty) \sim \rho_c^4/x^4$ for the quark and gluon correlators, respectively. Thus, the SI solution over the mathematical vacuum yields too slow asymptotics at large distances. We should conclude that in order to have a realistic model of vacuum correlators, the important effects of the instanton interaction with the long-wave vacuum configurations have to be included.

The key point in the picture of a realistic instanton vacuum is the interaction of pseudoparticles in the vacuum. In [19], the interaction of a SI with an arbitrary weak external field has been examined and dipole–dipole forces in a far separated instanton–anti-instanton system have been derived. Later, in [20], this background field has been interpreted as a field of large-scale QCD vacuum fluctuations, and the influence of the quark and gluon condensates on the instanton density has been considered. The main assumption of the instanton liquid models [12] is the dominance of the instanton component in the vacuum and in particular, that the gluon condensate is saturated by a weakly interacting instanton liquid. In deriving instanton ensemble properties the instanton–anti-instanton interaction at intermediate separation starts to play a crucial role in the stabilization of the liquid [13]. However, it turns out that the final result strongly depends on the field ansatz for an instanton–anti-instanton configuration [21,14]. Further, in all instanton–anti-instanton ansätze suggested the influence of the physical vacuum on an instanton profile function has not been taken into account and the profile has only power decreasing asymptotics, which contradicts the expectations concerning the vacuum field correlators. Moreover, it is known that the instanton liquid is not responsible for large-scale effects like confinement [14]. Another point is that the instanton density n_c in the instanton liquid models is normalized by the value of the gluon condensate $\langle : (\alpha_s/\pi)F^a_{\mu\nu}F^a_{\mu\nu} :\rangle = 0.012\,\mathrm{GeV}^4$ obtained in [1] from an analysis of the charmonium spectrum. More recently, in [22], a detailed analysis based on charmonium, bottonium and heavy-light mesons have lead to a twice larger value of the gluon condensate $\langle : (\alpha_s/\pi) F^a_{\mu\nu} F^a_{\mu\nu} : \rangle = 0.023\,\mathrm{GeV}^4$. Indefiniteness in the normalization provides a window for the existence of a large-scale field component in the QCD vacuum along with short-scale instantons.

In the present work, assuming dominance of the weak interacting instanton liquid in the QCD vacuum, we suggest that there is also a weak residual component of the vacuum field with a large correlation length R of the order of the confinement size. We are going to show, assuming only very general properties of a weak large-scale vacuum field, that it deforms an instanton at large distances, leading to exponentially decreasing asymptotics of the instanton profile and the instanton-induced vacuum field correlators. The vacuum model considered is a two-phase one. The large-scale phase is described by the background field and dominates at distances compared with the confinement size. The short-scale phase is dominated by instantons and is responsible for the spontaneous breaking of chiral symmetry and the solution of the $U_A(1)$ problem. The vacuum model suggested reveals a definite non-locality mechanism within the QCD framework. We shall illustrate this by analyzing the vacuum gluon correlation functions $D(x)$ and $D_1(x)$. Unfortunately, the normalization of contributions from different phases to the gluon condensate is not fixed by the instanton model and remains as a free parameter, but the form of the correlation functions can be described in detail. The latter is the main motivation of the present work. The determination of the vacuum field correlators is one of the main tasks of the theory in describing the non-perturbative dynamics at large distances compatible with the typical hadron size.

The paper is organized as follows: in Sect. 2 the instanton field in the background of weak large-scale vacuum fluctuations is considered; solutions of the vacuum field equations for small and large distances are analyzed separately, and the constrained instanton solution interpolating two asymptotics is suggested in an ansatz form. In Sect. 3, the space coordinate behavior of the non-local correlator of the gluon field strengths is found and the main asymptotics of the correlators $D(x)$, $D_1(x)$ at large distances are derived. These asymptotics are driven by the parameters of the large-scale vacuum fluctuations: by their strength and correlation length R.

2 Constrained instantons in QCD vacuum

The classical Yang–Mills equations in the Euclidean space

$$D_\mu F_{\mu\nu}(x) = 0, \tag{5}$$

where the covariant derivative is $D_\mu = \partial_\mu - \mathrm{i} A^a_\mu \tau_a/2$, have an instanton (+) ((−) is for an anti-instanton) as an (anti-) self-dual finite-action solution with topological charge $Q = \pm 1$,

$$A^{a,\pm}_{\mathrm{sing},\mu}(x; x_0) = 2 O^{ab}_I \eta^{b,\pm}_{\mu\nu} (x - x_0)_\nu \, \varphi^I_{\mathrm{g}}(x - x_0)$$
$$\text{with } \varphi^I_{\mathrm{g}}(x) = \frac{\rho^2}{x^2(x^2+\rho^2)} \text{ (in singular gauge)}, \tag{6}$$

localized in a size ρ. In (6), x_0 is the position of the instanton center, O^{ab}_I is the orthogonal matrix of the instanton's global orientation in the color space and $\eta^{a,\pm}_{\mu\nu} = \epsilon_{4a\mu\nu} \mp (1/2)\epsilon_{abc}\epsilon_{bc\mu\nu}$ are 't Hooft symbols. The solution (6) is written in the *singular* gauge within the $SU(2)$ subgroup (with generators $\tau_a/2$) of the $SU_{\mathrm{c}}(3)$ theory. This classical field configuration reflects the symmetries of the initial theory in terms of collective variables corresponding to translational transformations, rotational symmetry in color space and conform transformations.

The solution (6) is given in the mathematical vacuum and has the unpleasant property of a very slow decay at large distances as noted in the Introduction. This situation is inadequate since the physical vacuum is not empty but looks like a medium densely populated by large-scale vacuum field fluctuations. In the background of the large-scale fluctuations there have developed non-perturbative fluctuations of a smaller size among those instantons which play a dominating role. The long-wave gluon vacuum field, which is the background for a selected instanton, may be of a more general origin, and phenomenologically can be parameterized by the vacuum correlation functions of the gluon operators contributing to the corresponding non-local condensates.

What is important is that at a *random* (stochastic) background vacuum field with *fixed* vacuum expectation values of the singlet operators the scale invariance of the effective theory is spoiled already at the quasi-classical level and the instantons are *no longer exact solutions* of the field equations, and the Dirac operator has no zero modes.

A similar situation has been observed in the standard electroweak Yang–Mills–Higgs model. In this case, the background of Higgs field with non-zero vacuum value $\langle\varphi\rangle$ and coupling λ affects the instanton configuration [23]. In the presence of (even small) effects violating scale invariance of the initial theory, an instanton solution does not exist at all. Nevertheless, as was stated in [23] and fairly explained in [24–26], if the Higgs field is rather weak and some additional constraints are introduced, there may be constructed an approximate solution, a so-called constrained instanton (CI). These constraints limit the degree of freedom along the size ρ parameter. The constrained solution at small distances $|x| << \lambda^{1/2} \langle\varphi\rangle^{-1}$ approximately has the form of an instanton, and at large distances $|x| >> \lambda^{1/2} \langle\varphi\rangle^{-1} >> \rho$ has the asymptotics of a massive (gauge boson) particle $\exp(-g \langle\varphi\rangle |x|/\lambda^{1/2})$. In [24] it has also been noted that the gauge field propagator of the CIs decays exponentially at large $|x|$ and thus does not affect the long-range behavior of the theory.

The aim of this section is to show that an analogous phenomenon takes place in QCD on considering an instanton field $A^I_\mu(x)$ in the physical vacuum. In distinction with the Yang–Mills–Higgs model, in pure QCD there is no Higgs field from the beginning and it is the long-scale vacuum gluon field, $b_\mu(x)$, that models a source perturbing an instanton at large distances. The deep reason of this effect lies in the existence of the quantum anomaly in the trace of the energy-momentum tensor [27]. It will be shown that the presence of this background field, characterized by its vacuum expectation value $\langle (F^a_{b,\mu\nu})^2\rangle_{\rm b}$ and correlation length R (introduced below), sets the final scale and defines the deformation of the instanton solution in the asymptotic region $|x| \gtrsim [\langle (F^a_{b,\mu\nu})^2\rangle_{\rm b} R]^{-1/3} >> \rho$. Here and below, $\langle \cdots \rangle_{\rm b} = \int {\rm d}\sigma\,[b] \cdots$ means the average over non-perturbative random background field weighted with some measure ${\rm d}\sigma\,[b]$. This solution is stable against shrinking the instanton to a point if some constraints are added. By analogy with the solutions analyzed in the Yang–Mills–Higgs model [24], we shall call these interpolating fields constrained (or deformed) instanton solutions.

In the following, we analyze the vacuum field configuration of the single (constrained) instanton $A^{\rm CI}_\mu(x)$ of fixed size and orientation in the color space in the background of the large-scale topologically neutral random vacuum field $b_\mu(x)$[2]

$$A_\mu(x) = A^{\rm CI}_\mu(x, x_0) + b_\mu(x), \tag{7}$$

with the gauge transformation property

$$A_\mu(x) \to U^\dagger(x)\, A_\mu(x) U(x) + {\rm i} U^\dagger(x)\, \partial_\mu U(x), \tag{8}$$

where $U(x)$ is a gauge transformation matrix. When one deals with gauge-non-invariant objects, any convenient gauge can be chosen. Considering the field ansatz (7), one has to take the instanton field $A^{\rm CI}_\mu(x)$ in the *singular* gauge [13,14]

$$A^{{\rm CI},a,\pm}_{{\rm sing},\mu}(x) = 2 O^{ab}_I \eta^{b,\pm}_{\mu\nu}\,(x-x_0)_\nu\, \varphi^{\rm CI}_{\rm g}(x-x_0), \tag{9}$$
$$x^2 \varphi^{\rm CI}_{\rm g}(x)\big|_{x\to 0} = 1, \qquad \varphi^{\rm CI}_{\rm g}(x)\big|_{x\to\infty} = 0.$$

The last conditions mean that the constrained instanton has a finite action and a modulo unit topological charge, but, in general, ceases to be self-dual field. In the coordinate space the instantons in this specific gauge fall off rapidly enough to provide a weak interaction with the background field and the quasi-classical approach is justified. The weakness of the interaction allows us in the following to neglect the back reaction of the instanton on the background field. As to the non-perturbative background field, it is convenient to put it in the Fock–Schwinger gauge [28] with the fixed point coinciding with the instanton center due to translational invariance arguments:

$$(x - x_0)_\mu\, b^\mu(x) = 0. \tag{10}$$

In the following we put the center of the instanton x_0 to the origin of the coordinates, $x_0 = 0$.

As an illustrative model for the background field, one can keep in mind the self-dual homogeneous vacuum gluon field $b^a_\mu(x) = (1/2) n^a b_{\mu\nu} x_\nu$, $\;b_{\mu\nu} b_{\mu\nu} = b^2$, where n^a is the orientation vector in color space and $b_{\mu\nu}$ is the constant field-strength tensor [30]. The corresponding measure $\int {\rm d}\sigma\,[b] = \int_0^\infty {\rm d}b D(b) \int {\rm d}\Omega \int {\rm d}\Omega_{\rm c}$ averages over field amplitude and its orientation in the configuration and color spaces. This configuration with an infinite correlation length $R = \infty$ and an infinite topological charge quite correctly describes the situation at small and intermediate distances, comparable with the instanton size, but at larger distances the effect of the finite correlation length of the physical background becomes important. The introduction of a finite correlation length can be imagined as the inclusion of a domain structure in the vacuum [29]. This kind of considerations are basically the stochastic vacuum model [28]. In the absence of a consistent theoretical approach to the large distance dynamics one is lead to a phenomenological elaboration of the problem.

We assume that at small distances the CI field dominates and the background field $b_\mu(x)$ is regarded as a perturbation on $A^{\rm CI}_\mu(x)$. At distances much larger compared to the instanton size ρ, the background field $b_\mu(x)$ is still weak, but strong enough to deform and suppress the instanton field.

The field strength can be written as

$$F^a_{\mu\nu}[A^{\rm CI} + b] = F^{{\rm CI},a}_{\mu\nu} + F^a_{b,\mu\nu} + \Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right], \tag{11}$$

where $F^{{\rm CI},a}_{\mu\nu} \equiv F^a_{\mu\nu}[A^{\rm CI}]$, $F^a_{b,\mu\nu} \equiv F^a_{\mu\nu}[b]$ and

$$\Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right] = f_{abc}(A^{{\rm CI},b}_\mu b^c_\nu + A^{{\rm CI},c}_\nu b^b_\mu),$$

and the effective Euclidean action of the instanton in the *random* background field becomes

$$S_{\rm E} \approx \frac{1}{4g^2}\left\langle \int {\rm d}^4x\, \{F^{{\rm CI},a}_{\mu\nu} F^{{\rm CI},a}_{\mu\nu} + \Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right] \Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right]\}\right\rangle_{\rm b}. \tag{12}$$

In deriving these expression we have used the color-singlet properties of the large-scale vacuum on an average

$$\left\langle F^a_{b,\mu\nu}\right\rangle_{\rm b} = 0 \tag{13}$$

and average over the relative orientation of the instanton and background fields in the color space. We also neglected higher order interaction terms. These terms effectively will be accumulated in the form of constraints below and, what is important in the present consideration, they do not influence the asymptotics of the solution.

Similar to the Affleck analysis [24], we come to the conclusion that for a background field configuration $b_\mu(x)$ with a *fixed* non-zero condensate value no instanton solution exists. This can be seen from the rescaling $x \to ax$, $A^{\rm CI}_\mu(x) \to a^{-1} A^{\rm CI}_\mu(ax)$, $b_\mu(x) \to b_\mu(ax)$, preserving the finite vacuum average $\left\langle F^2_{b,\mu\nu}\right\rangle_{\rm b} = {\rm const}$, under which $S_{\rm E}$ transforms to

$$S_{\rm E} \to \frac{1}{4g^2}\left\langle \int {\rm d}^4x\, \{F^{{\rm CI},a}_{\mu\nu} F^{{\rm CI},a}_{\mu\nu} + a^{-2} \Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right] \Delta F^a_{\mu\nu}\left[A^{\rm CI}, b\right]\}\right\rangle_{\rm b} \tag{14}$$

If $A^{\rm CI}_\mu(x)$ is a stationary point, then ${\rm d}S_{\rm E}/{\rm d}a = 0$ and the action is minimized by an instanton of vanishing size.

[2] A similar model has been considered before in [37].

Thus, given any field configuration we can always rescale it to get a smaller action, except in the trivial case.

Now, let us consider the problem from the point of view of the equations of motion for the deformed instanton in the background of large-scale random vacuum fluctuations, which becomes

$$D_\mu^{ab}\left[A^{\rm CI}\right]F_{\mu\nu}^{{\rm CI},b} + f^{bac}f^{bkl}\left(A_\mu^{{\rm CI},k}\left\langle b_\mu^c b_\nu^l\right\rangle_{\rm b} - A_\nu^{{\rm CI},k}\left\langle b_\mu^c b_\mu^l\right\rangle_{\rm b}\right) = 0. \quad (15)$$

In the Fock–Schwinger gauge the background field has a representation in terms of its strength

$$b_\mu^a(x) = \int_0^1 {\rm d}\alpha\alpha F_{b,\nu\mu}^a(\alpha x)x^\nu, \quad (16)$$

and the bilinear field averages become

$$\left\langle b_\mu^a(x)b_\nu^b(x)\right\rangle_{\rm b} = \int_0^1 {\rm d}\alpha \times \int_0^1 {\rm d}\beta\alpha\beta x_\rho x_\sigma \left\langle F_{b,\rho\mu}^a(\alpha x)F_{b,\sigma\nu}^b(\beta x)\right\rangle_{\rm b}. \quad (17)$$

In the non-Abelian case the correlator in the integrand of (17) is not gauge invariant; however, in the Fock–Schwinger gauge this correlator coincides with the gauge-invariant correlator in which the field strengths are connected by the Schwinger phase factors $\widetilde{E}(\alpha x,0)\widetilde{E}(0,\beta x)$ in the adjoint representation. Thus, in this specific gauge the gauge-variant left side of (17) can be expressed in terms of an gauge-invariant quantity. Due to the gauge invariance the latter correlator admits a physically motivated model;

$$x_\rho x_\sigma\left\langle F_{\rho\mu}^a(\alpha x)F_{\sigma\nu}^b(\beta x)\right\rangle_{\rm b} = \frac{\delta^{ab}}{N_{\rm c}^2-1}\frac{\left\langle F_b^2\right\rangle_{\rm b}}{12}\left(x^2\delta_{\mu\nu}-x_\mu x_\nu\right)\widetilde{B}\left(z^2\right)\Big|_{z=x(\alpha-\beta)}, \quad (18)$$

where the function

$$\widetilde{B}\left(z^2\right) = \widetilde{D}\left(z^2\right)+\widetilde{D}_1\left(z^2\right)+z^2\partial\widetilde{D}_1\left(z^2\right)/\partial z^2 \quad (19)$$

is defined via the invariant functions $\widetilde{D}\left(z^2\right)$ and $\widetilde{D_1}\left(z^2\right)$ parameterizing the gauge-invariant two-point correlator (2) of the background field strengths, with normalization $\widetilde{D}(0)=\widetilde{\kappa}$, $\widetilde{D_1}(0)=1-\widetilde{\kappa}$. The contribution of the background field to the gluon condensate is denoted by $\left\langle F_b^2\right\rangle_{\rm b}$. With these definitions the equations of motion of the CI field interacting with a random large-scale vacuum fluctuation field can be cast in the form

$$D_\mu^{ab}\left[A^{\rm CI}\right]F_{\mu\nu}^{{\rm CI},b}(x) - \frac{N_{\rm c}\left\langle F_b^2\right\rangle_{\rm b}}{24(N_{\rm c}^2-1)}x^2\Phi\left(x^2\right)A_\mu^{{\rm CI},a}(x) = 0, \quad (20)$$

where

$$\Phi\left(x^2\right) = 4\int_0^1{\rm d}\alpha\int_0^1{\rm d}\beta\alpha\beta\widetilde{B}\left[(\alpha-\beta)^2x^2\right], \quad \Phi(0)=1, \quad (21)$$

and $N_{\rm c}$ is the number of colors.

Let us discuss the properties of the solution of (20). In the absence of the background field $\left\langle F_b^2\right\rangle_{\rm b}=0$ there exists an instanton solution (6). For $\left\langle F_b^2\right\rangle_{\rm b}$ small enough, such that $\left\langle F_b^2\right\rangle_{\rm b}\ll 1/\rho^4$, we should expect to find a solution of (20) in perturbation theory in the small parameter $\left\langle F_b^2\right\rangle_{\rm b}\rho^4$, which reduces to (6) when $\left\langle F_b^2\right\rangle_{\rm b}\to 0$. However, such a perturbative solution does not exist. The reason is that for the higher order perturbation terms appropriate finite-action boundary conditions at large distances cannot be enforced [24].

The operators that act on higher order terms possess a zero mode $\partial A_\mu^{\rm CI}/\partial\rho$:

$$\nabla_\mu\left(\nabla_\mu\frac{\partial A_\nu^{\rm CI}}{\partial\rho}-\nabla_\nu\frac{\partial A_\mu^{\rm CI}}{\partial\rho}\right)+{\rm i}\left[\frac{\partial A_\mu^{\rm CI}}{\partial\rho},F_{\mu\nu}\left[A^{\rm CI}\right]\right]=0, \quad (22)$$

which determines a priori the behavior of perturbations around instanton terms at infinity. A way of getting around this difficulty [24] is to extremize the action $S_{\rm E}$, see (12), subject to a constraint. The choice of an explicit form of the constraint is quite arbitrary. In [24] has been proposed a global constraint of the general form

$$C_{\rm constr}^{nl}(A) = \int{\rm d}^4x\left[O_d(A)-O_d\left(A^{\rm CI}\right)\right]=0,$$

where the gauge-invariant local operator $O_d(A)$ has a canonical dimension $d>4$. The relevant stationary configuration will be a solution of the equations of motion (20) but with the constraint term added into the right-hand side

$$\left.\frac{\delta S_{\rm E}(A)}{\delta A_\mu(x)}\right|_{A_\mu^{\rm CI}} = \sigma\left.\frac{\delta C_{\rm constr}(A)}{\delta A_\mu(x)}\right|_{A_\mu^{\rm CI}}. \quad (23)$$

The Lagrange multiplier σ in (23) is to be determined order by order in perturbative theory in $\left\langle F_b^2\right\rangle_{\rm b}\rho^4$, which provides the correct boundary conditions for the higher order terms. The constrained instanton $A^{\rm CI}$ is the unique solution of (23) obtained by this procedure, the $A^{\rm CI}$ solution goes to (6) when $\left\langle F_b^2\right\rangle_{\rm b}\rho^4\to 0$. Unfortunately, this kind of constraints is nonlinear and the higher order terms in $\left\langle F_b^2\right\rangle_{\rm b}\rho^4$, depending on the constraint, are difficult to evaluate in practice.

Another way has been suggested in [26], where a linear constraint of the general form

$$C_{\rm constr}^{l}(A) = \int{\rm d}^4x{\rm tr}\left\{\left(A_\mu(x)-A_\mu^{\rm CI}(x)\right)f_\mu^\rho(x)\right\} = 0 \quad (24)$$

has been introduced. It has been proposed, [26], instead of fixing the constraint to solve the equation for $A_\mu^{\rm CI}(x)$, which is almost an impossible task, to choose $A_\mu^{\rm CI}(x)$ first and then find the constraint $f_\mu^\rho(x)$ itself by substituting $A_\mu^{\rm CI}(x)$ into the left-hand side of the constrained equation (23). In this way, the freedom in choosing the constraint can be used to find it by a given solution.

One can also restrict oneself by considering only the local operators defining constraints that fall at infinity more rapidly than the interaction term in (20). Under this condition, it is easy to obtain the behavior of the instanton enveloped in the background field at distances far from the instanton center. This large-distance asymptotics of CI, like its behavior at small distances $A_\mu^I(x;\rho)$, is a constraint-independent part of the solution.

We are interested in the asymptotic behavior of the function $\Phi\left(x^2\right)$, see (21), where the interaction term becomes dominant over the instanton self-interaction. It is nice that the leading asymptotics of $\Phi\left(x^2\right)$, $\Phi\left(x^2\right) \sim R/|x|$, is independent of a particular form for the function $\widetilde{B}(z^2)$, since it has a specific argument dependence $z^2 = (\alpha-\beta)^2 x^2$. Let us illustrate this property using a few natural ansätze for this function

$$\begin{aligned} \widetilde{B}_M\left(x^2\right) &= R^2/\left(x^2+R^2\right), \quad \text{(monopole)} \\ \widetilde{B}_E\left(x^2\right) &= \exp(-|x|/R), \quad \text{(exponential)} \\ \widetilde{B}_G\left(x^2\right) &= \exp(-x^2/R^2), \quad \text{(Gaussian)} \end{aligned} \tag{25}$$

that yield, respectively, (see Appendix B)

$$\Phi^{as}\left(x^2\right) \overset{|x|\to\infty}{\to} \frac{8}{3} a_\Phi \frac{R}{|x|}, \quad \text{where } a_\Phi = O(1) = \begin{cases} \frac{\pi}{2} & \text{monopole,} \\ 1 & \text{exponential,} \\ \frac{\sqrt{\pi}}{2} & \text{Gaussian.} \end{cases} \tag{26}$$

The exponential ansatz (modulo powers) appears as an asymptotic expression of the solution of the Yang–Mills–Higgs model [24,25] and also is used in the parameterization of the large-scale behavior of lattice QCD data [7]. The monopole form resembles the asymptotic behavior of the SI correlator [15], where, in fact, it has a steeper decrease like $1/x^4$.

The asymptotic behavior of the instanton solution deformed by large-scale vacuum fluctuations at large Euclidean distances $|x| \to \infty$ can be derived from the analysis of the equations

$$\partial_\mu(\partial_\mu A_\nu^{\mathrm{CI}} - \partial_\nu A_\mu^{\mathrm{CI}}) - \eta_{\mathrm{g}}^3 |x| A_\mu^{\mathrm{CI}} = 0, \tag{27}$$

with

$$\eta_{\mathrm{g}} = \left(\frac{a_\Phi N_{\mathrm{c}}}{9\left(N_{\mathrm{c}}^2-1\right)} R \left\langle F_b^2\right\rangle_{\mathrm{b}}\right)^{1/3}.$$

Due to the decreasing character of the field asymptotics, $A_\mu^{\mathrm{CI}} \to 0$ at $|x| \to \infty$, only linear terms in the short-wave CI field A_μ^{CI} are kept in the kinetic part of equation (27)[3].

[3] At this point we have to note that the influence of the instanton ensemble on the instanton profile has been discussed in [13]. The authors found that this interaction perturbs the self-interaction part by the term $\mu_{\mathrm{g}}^2 A_\mu$. It turns out that numerically the coefficient μ_{g}^2 strongly depends on the instanton–anti-instanton ansatz chosen [21] and, as is seen from (27), has a subleading behavior in the limit of large x.

For the profile function $\varphi_{\mathrm{g}}^{as}(x^2)$ (defined by $\overline{\eta}_{\nu\mu}^a \equiv \eta_{\nu\mu}^{+a}$ in the following) we have

$$A_\mu^{\mathrm{CI},a}(x) = \overline{\eta}_{\nu\mu}^a \frac{x_\nu}{x^2} \varphi_{\mathrm{g}}^{as}\left(x^2\right), \tag{28}$$

we find from the asymptotic equation (27) the large-distance solution

$$\varphi_{\mathrm{g}}^{as}\left(x^2\right) \sim K_{4/3}\left[\frac{2}{3}\left(\eta_{\mathrm{g}}|x|\right)^{3/2}\right], \tag{29}$$

where $K_\nu(z)$ is the modified Bessel function with index ν having the asymptotic behavior $K_\nu(z) \to (\pi/2z)^{1/2}\mathrm{e}^{-z}$ as $z \to \infty$. We have to note that in the case of the homogeneous background field with infinite correlation length one gets an equation similar to (27), but with the coefficient proportional to x^2 in the last term, which results in a Gaussian form of the asymptotics [27].

The weakness of the instanton–background interaction allows us to assume that the dimensionless parameter $\alpha_{\mathrm{g}} \equiv \eta_{\mathrm{g}}\rho$ is small. This means that the region where the instanton field dominates (small distances), and the region where the background field dominates (large distances), are well separated and the large distance effects do not destroy the instanton. Then, the overall constant is determined by matching at distances $\rho \ll |x| \ll \eta_{\mathrm{g}}^{-1}$ the leading terms of the expansions of $A_\mu^{\mathrm{CI}}(x)$ at small distances, which is an instanton $A_\mu^I(x)$, and at large distances, which is an asymptotic term; see (28) and (29). We have

$$A_\mu^{\mathrm{CI},a}(x) = \overline{\eta}_{\nu\mu}^a \frac{2x_\nu}{x^2} a_{4/3}\alpha_{\mathrm{g}}^2 K_{4/3}\left[\frac{2}{3}\left(\eta_{\mathrm{g}}|x|\right)^{3/2}\right], \quad \text{where } a_{4/3} = \frac{2}{\Gamma(1/3)\, 3^{1/3}} \tag{30}$$

is the normalization coefficient and $\Gamma(z)$ is the Gamma function.

Thus, we find that in the singular gauge the CI solution has an exponential decreasing character (30) far from the instanton center. This behavior sharply differs from the asymptotics decreasing with a power of SI (6). This modification of the behavior follows from the fact that the instanton solution is considered in the physical vacuum populated by large-scale gluon field fluctuations. The background field modifies the long-distance behavior of the instanton and leads to the appearance of the "second scale" parameter $\Lambda_{\mathrm{g}} = 1/\eta_{\mathrm{g}}$ [15] in the gluon distributions. At the same time, the effect of long-wave vacuum fluctuations is not very essential for the behavior of the instanton at short distances.

Now we are looking for the constrained solution of (20) in the ansatz form

$$A_\mu^{CIa}(x) = 2\overline{\eta}_{\nu\mu}^a x_\nu \varphi_{\mathrm{g}}\left(x^2\right), \tag{31}$$

which has the following behavior at small and large distances:

$$\varphi_{\mathrm{g}}\left(x^2\right) = \frac{1}{x^2}$$

$$\cdot\begin{cases} \frac{\rho^2}{(x^2+\rho^2)} + (\eta_{\mathrm{g}}\,|x|)^3\,\varphi_{(1)}\left(\frac{\rho}{|x|}\right)\cdots \\ \text{at } |x|\to 0, \\ a_{4/3}\alpha_{\mathrm{g}}^2 K_{4/3}\left[\frac{2}{3}(\eta_{\mathrm{g}}\,|x|)^{3/2}\right] + \left(\frac{\rho^2}{x^2}\right)^2 \varphi^{(1)}(\eta_{\mathrm{g}}\,|x|)\cdots \\ \text{at } |x|\to\infty, \end{cases} \quad (32)$$

where the forms of the expansions can be deduced from expanding the leading terms with respect to $\rho/|x|$ and $\eta_{\mathrm{g}}|x|$, respectively. The first terms in the expansions are exact and constraint-independent ones; however, as was shown in [24], all higher order terms are dependent on the constraint chosen.

The condition of finiteness of the action, which is also a constraint-independent one, should be imposed on the desirable solution. To reach this goal, let us rewrite the CI field strength as

$$F_{\mu\nu}^{CIa}(x) = 4\left[\overline{\eta}_{\mu\nu}^a \omega_1(x) + \left(x_\mu \overline{\eta}_{\nu\rho}^a - x_\nu \overline{\eta}_{\mu\rho}^a\right) x_\rho \omega_2(x)\right] \quad (33)$$

with

$$\omega_1(x) = x^2\varphi_{\mathrm{g}}^2(x^2) - \varphi_{\mathrm{g}}(x^2),$$
$$\omega_2(x) = \varphi_{\mathrm{g}}^2(x^2) + \frac{\partial\varphi_{\mathrm{g}}(x^2)}{\partial x^2}, \quad x_\rho^2 = x^2+\rho^2. \quad (34)$$

to be used below. With this parameterization we have the action

$$S_{\mathrm{E}}^{\mathrm{CI}} = \frac{1}{4g^2}\int \mathrm{d}^4x\,\left[F_{\mu\nu}^a(x)\,F_{\mu\nu}^a(x)\right]^{\mathrm{CI}} \quad (35)$$

with the action density

$$\left[F_{\mu\nu}^a(x)\,F_{\mu\nu}^a(x)\right]^{\mathrm{CI}} = 96\left[\omega_1^2(x) + \omega_3^2(x)\right], \quad (36)$$

where

$$\omega_3(x) = x^2\omega_2(x) - \omega_1(x).$$

Now, if we use the singular gauge for A_μ^{CI}, then, to guarantee finiteness of the action, the following condition has to be fulfilled:

$$x^2\varphi_{\mathrm{g}}(x^2)\,|_{x^2\to 0} \to 1 + O(x^2). \quad (37)$$

For further reference we here present the well-known expressions for the SI profiles in the singular and regular gauges:

$$\varphi_{\mathrm{g}}^{\mathrm{sing},I}(x^2) = \frac{\rho^2}{x^2x_\rho^2}, \quad \varphi_{\mathrm{g}}^{\mathrm{reg},I}(x^2) = \frac{1}{x_\rho^2}, \quad (38)$$

$$\omega_1^{\mathrm{sing},I}(x) = -\frac{\rho^2}{\left(x_\rho^2\right)^2}, \quad x^2\omega_2^{\mathrm{sing},I}(x) = 2\omega_1^{\mathrm{sing},I}(x), \quad (39)$$

$$\omega_1^{\mathrm{reg},I}(x) = -\frac{\rho^2}{\left(x_\rho^2\right)^2}, \quad \omega_2^{\mathrm{reg},I}(x) = 0, \quad (40)$$

and gauge-independent expressions for the density action and the action itself:

$$\left[F_{\mu\nu}^a(x)\,F_{\mu\nu}^a(x)\right]^I = \frac{192\rho^4}{\left(x_\rho^2\right)^4}, \qquad S_{\mathrm{E}}^I = \frac{8\pi}{g^2}. \quad (41)$$

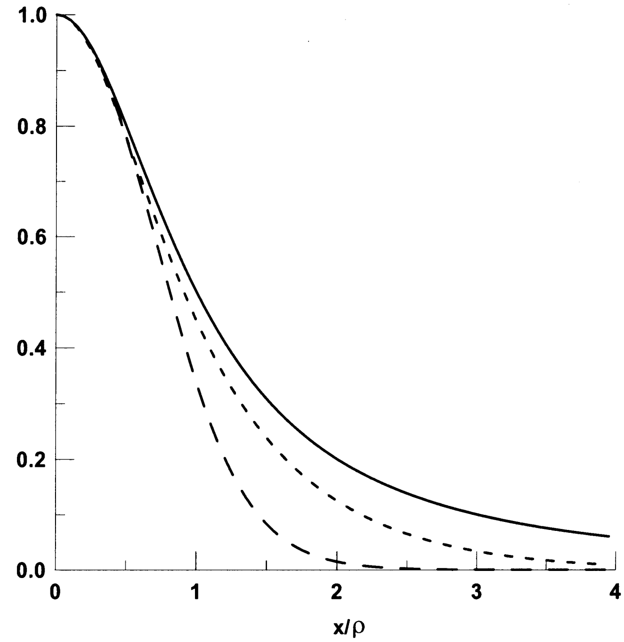

Fig. 1. The constrained instanton profile functions $x^2\varphi_{\mathrm{g}}(|x|/\rho)$ (31), corresponding to the ansatz (44), at different values of the parameter $(\rho\eta_{\mathrm{g}})^2$: $(\rho\eta_{\mathrm{g}})^2 = 0$ solid line (instanton case), $(\rho\eta_{\mathrm{g}})^2 = 0.5$ short dashed line, $(\rho\eta_{\mathrm{g}})^2 = 3$ long dashed line

By using the asymptotic properties of the CI solution (32) and the finite-action condition (37), which are constraint independent, we are able to construct the ansatz. Certainly, this procedure is not unique and in principle one can impose further physical requirements to constrain the behavior of the solution in the intermediate region. These details, however, can be taken into account by choosing proper constraints. Thus, the freedom in choosing the constraint can be used to find it by a given solution, instead of solving complicated equations [26].

Let us consider the following ansätze for the CI profile written in the singular gauge

$$\varphi_1(x^2) = \frac{1}{x^2}\frac{K_{4/3}[z_{\rho,x}]}{K_{4/3}[z_{\rho,0}]}, \quad (42)$$

$$\varphi_2(x^2) = \frac{\overline{\rho}^2(x^2)}{x^2x_\rho^2}, \quad (43)$$

$$\varphi_3(x^2) = \frac{\overline{\rho}^2(x^2)}{x^2\left(x^2+\overline{\rho}^2(x^2)\right)}, \quad (44)$$

where we have introduced the notation

$$z_{\rho,x} = \frac{2}{3}\eta_{\mathrm{g}}^{3/2}\left(x^2+\rho^2\right)^{3/4},$$
$$\overline{\rho}^2(x^2) = a_{4/3}\alpha_{\mathrm{g}}^2 x^2 K_{4/3}[z_{0,x}], \qquad \overline{\rho}^2(0) = \rho^2.$$

Note that all ansätze have easily identifiable instanton parameters. By translational invariance the center of CI can be shifted in (42)-(44) from the origin to an arbitrary position x_0: $x\to x-x_0$. The CI profile functions corresponding to these ansätze are shown in Figs. 1 and 2 along with the

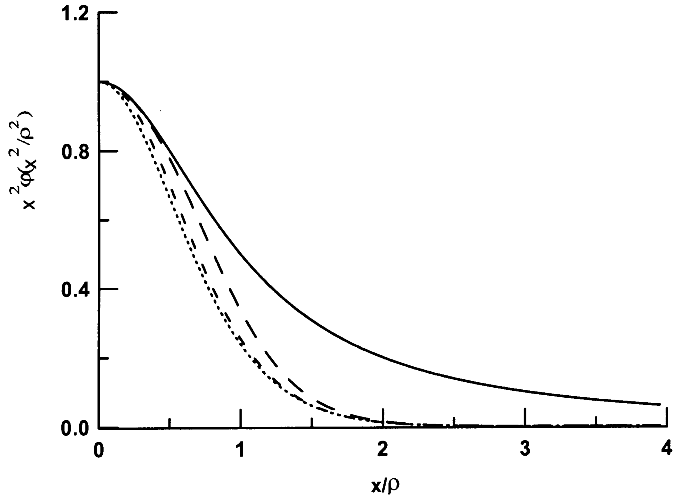

Fig. 2. The instanton (solid line) and constrained instanton profile functions $x^2\varphi_{\rm g}(|x|/\rho)$ (31) corresponding to different ansätze: (42) dotted line, (43) short dashed line, (44) long dashed line, at the large value of the parameter $(\rho\eta_{\rm g})^2 = 3$

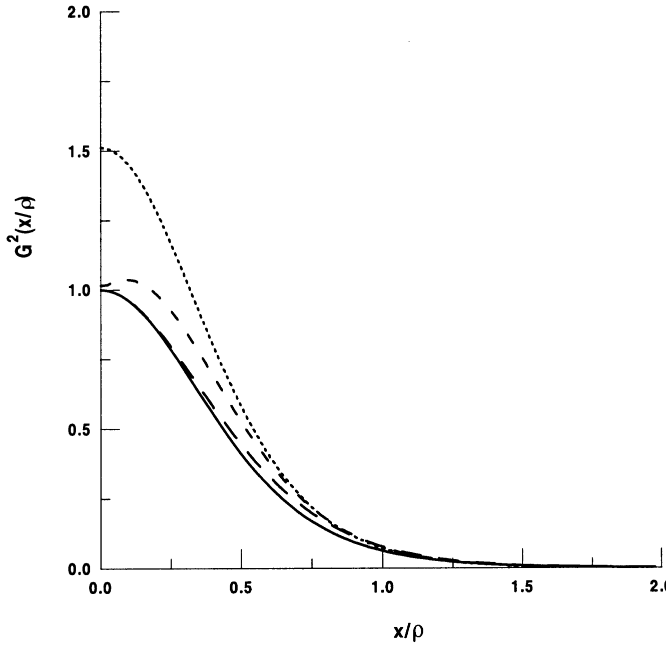

Fig. 3. The density action of the instanton (solid line) and constrained instanton $G^2(x^2) \equiv \rho^4/192 F^{Ia}_{\mu\nu}(x) F^{Ia}_{\mu\nu}(x)$, see (36), as a function of $|x|/\rho$ corresponding to different ansätze: (42) dotted line, (43) short dashed line, (44) long dashed line, at the value of the parameter $(\rho\eta_{\rm g})^2 = 1$

instanton profile (38). Figures 1 and 2 display the dependence of the profile function on the external field parameter $\rho\eta_{\rm g}$ and on the form of the ansatz, respectively. To make the difference more clear, we also take for illustration a large value of the parameter $(\rho\eta_{\rm g})^2 = 3$. We see that if at small distances the CI is close to the instanton form, then at large distances this solution has an exponential asymptotical behavior instead of a power-like one for the instanton.

The last two ansätze are similar to the ones suggested in [26], where preference has been given to the $\varphi_3(x^2)$ form, since it has better convergence properties in the expansion of CI; see (32). Moreover, with this profile the constrained solution in the regular gauge looks similar to the SI case

$$\varphi_3^{\rm reg,CI}(x^2) = \frac{1}{x^2 + \bar{\rho}^2(x^2)}.$$

In order to pass from the constraint instanton in a singular gauge to the instanton in a regular gauge one can translate the general gauge transformation (8) into the form

$$A^{\rm CI}_\mu(x) \to \Omega^\dagger(x) A^{\rm CI}_\mu(x)\Omega(x) + {\rm i}\Omega^\dagger(x)\partial_\mu\Omega(x), \quad (45)$$

$$b_\mu(x) \to \Omega^\dagger(x) b_\mu(x)\Omega(x),$$

with the transformation matrix

$$\Omega(x) = \frac{{\rm i}\tau^-_\mu x_\mu}{|x|}.$$

We have numerically calculated the dependence of the CI classical action, (35) and (36), on the instanton size ρ. This dependence for the three ansätze (42), (43) and (44) is shown in Figs. 3 and 4. We have to stress that the profiles of the field $A^{\rm CI}_\mu$ and the action $S^{\rm CI}_{\rm E}$ depend on the choice of constraint. However, the full effective action, with the terms coming from the Jacobian included, is constraint independent [26]. The weak dependence of the action, $S^{\rm CI}_{\rm E}$, on the profiles $\varphi_{1,2,3}(x^2)$ (see Fig. 4) indicates that in the regions of the parameters $\rho\eta_{\rm g} \lesssim 1$, the influence of these additional terms is small and the exponential part of the action, $S^{\rm CI}_{\rm E}$, can be used as a good approximation. We see in Fig. 4 that the CI action is larger than the instanton one and monotonically grows with the instanton size. It is natural because the CI "solution" is not self-dual and does not realize the minimum of the action. Instead, it represents the bottom of the valley parameterized by the quasi-zero mode ρ.

We are not going to discuss further details in the construction of the total effective CI action that takes into account small quantum oscillations around the non-perturbative configuration (7) and the interaction of constrained instantons, and postpone these for a further publication. Just let us point out that other effects dominating the effective action at small ρ come from the running coupling constant $g^2(\rho)$ and the path integral measure over the size of the instanton ${\rm d}\rho/\rho^5$. It is well known that in the model of the coupling constant, which freezes it to a constant at some large ρ_0, the corrected action

$$S^{\rm TOT}_{\rm E}(\rho) = \frac{1}{4g^2(\rho)}\int {\rm d}^4x \left[F^a_{\mu\nu}(x) F^a_{\mu\nu}(x)\right]^{\rm CI} + 5\ln\rho,$$

as a function of the instanton size, has a minimum. The position of the minimum is correlated with the freezing parameter ρ_0, which can be chosen to provide the value $\rho_{\rm min} \approx 2\,{\rm GeV}^{-1}$, and the environment of the large-scale vacuum fluctuations makes the minimum more prominent

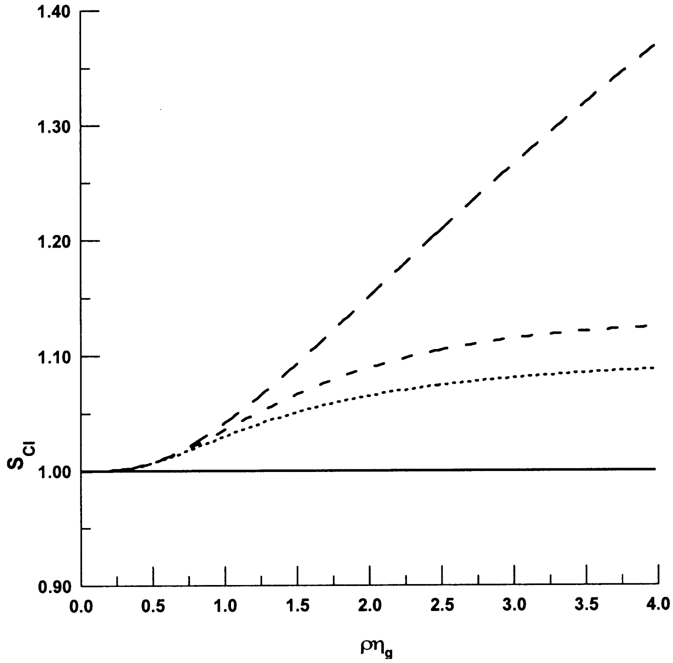

Fig. 4. The classical action of the instanton (solid line) and constrained instanton $S_{\rm CI}$, (35), as a function of $\rho\eta_{\rm g}$ corresponding to different ansätze: (42) dotted line, (43) short dashed line, (44) long dashed line. The action is given in units of $8\pi^2/g^2$

(see [31] for a recent discussion of similar results). However, the typical CI action is rather large, the numerical value being around 25. This means that the configurations with a small number of instantons and anti-instantons are not important statistically and this suggests that the interacting instanton and anti-instanton ensemble could be a more important type of configuration. The leading interaction term of a widely separated instanton–anti-instanton pair in the physical vacuum as described by (42)–(44) falls with separation L like $\exp(-4/3(\eta_{\rm g}L)^{3/2})$ and differs from the power-like decreasing behavior found in [19,13] in the unconstrained case. In the following, considering the non-local properties of a gluon condensate, we accept that the instanton liquid is formed due to the instanton–anti-instanton interaction and the instanton density $n_{\rm c}$ and size $\rho_{\rm c}$ are fixed [14]: $n_{\rm c}\approx 1\,{\rm fm}^{-4}$, $\rho_{\rm c}\approx 1/3\,{\rm fm}$.

3 Short-range vacuum correlators in the constrained instanton model

Within the model considered, the full gluon correlator may be conventionally written by using the expression for the field strength (11) as

$$\begin{aligned}&\langle: F_{\mu\nu}[A^{\rm CI}+b]\,(x)\,F_{\rho\sigma}[A^{\rm CI}+b]\,(y) :\rangle\\ &= \left\langle F^{\rm CI}_{\mu\nu}\,(x)\,F^{\rm CI}_{\rho\sigma}\,(y)\right\rangle + \langle: F_{b,\mu\nu}\,(x)\,F_{b,\rho\sigma}\,(y) :\rangle\\ &\quad + \left\langle: \varDelta F_{\mu\nu}[A^{\rm CI},b]\,(x)\,\varDelta F_{\rho\sigma}[A^{\rm CI},b]\,(y) :\right\rangle^{\rm interf}, \quad (46)\end{aligned}$$

where the brackets $\langle\rangle$ mean averaging over vacuum fluctuations (7) and we do not display Schwinger phase factors explicitly. The last term represents the interference of short- and large-scale fields and will be discussed below. For the large-scale correlator (19) we have already suggested the model for the invariant function $\widetilde{B}(x^2)$ in (25). Now, let us calculate the short-range part of the gluon correlator.

Let us construct the correlator $D^{\mu\nu,\rho\sigma}(x-y)$ of the gluonic strengths (1) in the quasi-classical approximation by using the CI solutions given by (31) and (42)–(44). We will use a reference frame in which the instanton sits at the origin and one has a relative coordinate $(x-y)^\mu$ with respect to the position of the instanton center that is parallel to one of the coordinate axes, say $\mu=4$, this one serving as a "time" direction (i.e., $\vec{x}-\vec{y}=0, x_4-y_4=|x-y|$), and reduce the path-ordered exponential to an ordinary exponential

$$\hat{E}(x,y) = P\exp\left({\rm i}\int_x^y A^{\rm CI}_\mu(z){\rm d}z^\mu\right) = L^\dagger\,(x)\,L\,(y) \quad (47)$$

with

$$\begin{aligned}L\,(x) &= \exp\left(\mp{\rm i}\,\vec{\tau}\,\frac{\vec{x}}{\left|\vec{x}\right|}\alpha\left(\left|\vec{x}\right|,x_4\right)\right)\\ &= \mp{\rm i}\tau^\pm_\mu\cdot\widetilde{x}^\mu\,(x)\,, \quad (48)\end{aligned}$$

where

$$\alpha\left(\left|\vec{x}\right|,x_4\right) = \left|\vec{x}\right|\int_0^{x_4}{\rm d}t\varphi_{\rm g}\left(\left|\vec{x}\right|^2+t^2\right), \quad (49)$$

$$\begin{aligned}\tau^\pm &= (\mp{\rm i},\tau), \widetilde{x}^0\,(x) = \cos\alpha\,(x)\,,\\ \widetilde{x}^i\,(x) &= \left(x^i/\left|\vec{x}\right|\right)\sin\alpha\,(x)\,. \quad (50)\end{aligned}$$

The factor $L\,(x)$ coming from the Schwinger exponent can be accumulated in the definition of the field. This representation of the field may be called the *axial* gauge representation $A_\mu(z)n^\mu=0$, since in this gauge with the vector $n_\mu=x_\mu-y_\mu$ the Schwinger factor is $\hat{E}(x,y)=1$.

In the CI background the bilocal gluon correlator acquires the form

$$\begin{aligned}D^{\mu\nu,\rho\sigma}(x) &= \langle: {\rm Tr}\left(F_{(ax)\mu\nu}(0)F_{(ax)\rho\sigma}(x)\right) :\rangle\\ &= \sum_\pm n^\pm_{\rm c}\int{\rm d}^4z \quad (51)\\ &\quad\times\int{\rm d}\varOmega\,{\rm Tr}\left(F^\pm_{(ax)\mu\nu}(z-\frac{x}{2})F^\pm_{(ax)\rho\sigma}\left(z+\frac{x}{2}\right)\right),\end{aligned}$$

where $n^\pm_{\rm c}$ is the effective instanton/anti-instanton density, z is the collective coordinate of the instanton center and $\varOmega$ is its color space orientation.

To extract invariant functions $D\left(x^2\right)$ and $D_1\left(x^2\right)$ it is easier first to average over the instanton orientations in the color space and take the trace over color matrices by using the relations

$$\int \mathrm{d}\Omega O_b^a O_d^{+c} = \frac{1}{N_\mathrm{c}} \delta_d^a \delta_b^c, \qquad (52)$$

$$\tau_\mu^\pm \tau_\nu^\mp = \delta_{\mu\nu} + i\eta_{\mu\nu}^{a,\mp}\tau^a, \qquad \tau^a\tau^b = \delta^{ab} + i\varepsilon^{abc}\tau^c.$$

Then it is convenient to define the combinations of functions $D\left(x^2\right)$ and $D_1\left(x^2\right)$ [32]

$$A\left(x^2\right) = \delta_{\mu\rho}\delta_{\nu\sigma}\frac{D^{\mu\nu,\rho\sigma}(x)}{\left\langle 0\left|F_{\mu\nu}^2\right|0\right\rangle^{\mathrm{CI}}} = D\left(x^2\right) + D_1\left(x^2\right) + \frac{1}{2}x^2\frac{\partial D_1\left(x^2\right)}{\partial x^2}, \qquad (53)$$

$$B\left(x^2\right) = 4\frac{x_\mu x_\rho}{x^2}\delta_{\nu\sigma}\frac{D^{\mu\nu,\rho\sigma}(x)}{\left\langle 0\left|F_{\mu\nu}^2\right|0\right\rangle^{\mathrm{CI}}} = D\left(x^2\right) + D_1\left(x^2\right) + x^2\frac{\partial D_1\left(x^2\right)}{\partial x^2},$$

taking the boundary condition $D(0) + D_1(0) = 1$ and the asymptotic conditions $D(\infty) = D_1(\infty) = 0$. After direct, but lengthy calculations we come to the expressions for the form factors A and B:

$$\begin{aligned} A(x^2) &= \frac{8}{\pi}N_D\int_0^\infty \mathrm{d}r r^2 \\ &\times \int_0^\infty \mathrm{d}t\,\Big\{\left[\omega_1(z_+)\,\omega_1(z_-) + \omega_3(z_+)\,\omega_3(z_-)\right] \\ &\times \left(3 - 4\sin^2(\alpha_z)\right) \\ &-2\omega_2(z_+)\,\omega_2(z_-) \\ &\times \left[r^2x^2\left(1 - 2\sin^2(\alpha_z)\right) - rx\,(z_+\cdot z_-)\sin(2\alpha_z)\right]\Big\}, \end{aligned} \qquad (54)$$

$$\begin{aligned} B(x^2) &= \frac{16}{\pi}N_D\int_0^\infty \mathrm{d}r r^2 \\ &\times \int_0^\infty \mathrm{d}t\,\Big\{\omega_1(z_+)\,\omega_1(z_-)\left(3 - 4\sin^2(\alpha_z)\right) \\ &-\omega_1(z_+)\,\omega_2(z_-) \\ &\times \left[z_-^2 + 2t_-^2\left(1 - 2\sin^2(\alpha_z)\right) + 2rt_-\sin(2\alpha_z)\right] \\ &-\omega_2(z_+)\,\omega_1(z_-) \\ &\times \left[z_+^2 + 2t_+^2\left(1 - 2\sin^2(\alpha_z)\right) - 2rt_+\sin(2\alpha_z)\right] \\ &+\omega_2(z_+)\,\omega_2(z_-) \\ &\times \left[z_+^2z_-^2 + 2t_+t_-\,(z_+\cdot z_-)\right. \\ &\times \left.\left(1 - 2\sin^2(\alpha_z)\right) + 2rxt_+t_-\sin(2\alpha_z)\right]\Big\}, \end{aligned} \qquad (55)$$

where $z_\pm = (r, t_\pm)$, $t_\pm = t \pm x/2$, the forms $\omega_1(z)$, $\omega_2(z)$, and $\omega_3(z)$ are defined in (34), N_D is the normalization factor

$$N_D^{-1} = 6\int_0^\infty \mathrm{d}yy^3\left(\omega_1^2(y) + \omega_3^2(y)\right), \qquad (56)$$

and the phase factor

$$\alpha_z = r\int_{-\frac{x}{2}}^{\frac{x}{2}} \mathrm{d}\tau\varphi_\mathrm{g}\left(r^2 + (t+\tau)^2\right),$$

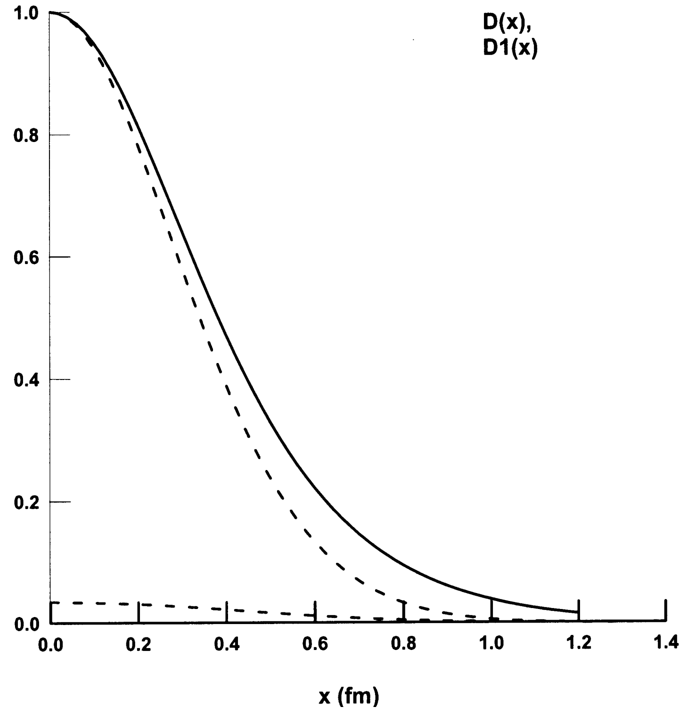

Fig. 5. The invariant functions D (top lines) and D_1 (bottom lines) (all normalized by $D(0)$) versus physical distance x, for the instanton size $\rho = 0.3\,\mathrm{fm}$ and parameters $(\rho\eta_\mathrm{g})^2 = 0$ (solid lines) and $(\rho\eta_\mathrm{g})^2 = 1$ (dashed lines)

reflects the presence of the $\hat{E}$ exponent in the definition of the bilocal correlator. These expressions for the field-strength correlators are general for any field given in the form (31). The gauge invariance of the form factors $A(x^2)$ and $B(x^2)$ can explicitly be checked for (54) and (55) by transforming, for example, the field A_μ from the singular to the regular gauges, see (45). The expressions for $A(x^2)$ and $B(x^2)$ may be considered as generating functions to obtain condensates of higher dimensions in the instanton model approach. From a technical point of view this procedure is more convenient than their direct calculations [32,15].

In the SI approximation the form factors $B^I(x^2) = A^I(x^2)$ reproduce the expression, see (21) from [15], for the gauge-invariant correlator. As has been shown in [15] (see also [34]), in the SI approximation the term with the second Lorentz structure, $D_1(x^2)$, parameterizing the gluon correlator (2) does not appear. This fact is due to the specific topological structure (self-duality) of the instanton solution. Both Lorentz structures arise in the r.h.s. of (51) if one takes into account the background fields.

The invariant functions $D(x^2)$ and $D_1(x^2)$ are determined numerically by solving (53) and are plotted in Fig. 5 in coordinate space and in Fig. 6 in momentum space. The constant κ defining the relative weight of the D functions ($D(0) = \kappa, D_1(0) = 1 - \kappa$) depends on the background field-strength parameter $\eta_\mathrm{g}\rho$ and is close to one in the region of reasonable physical parameters. It is equal to $\kappa = 1$ at $(\eta_\mathrm{g}\rho)^2 = 0$ (SI case), $\kappa = 0.997$ at $(\eta_\mathrm{g}\rho)^2 = 0.1$

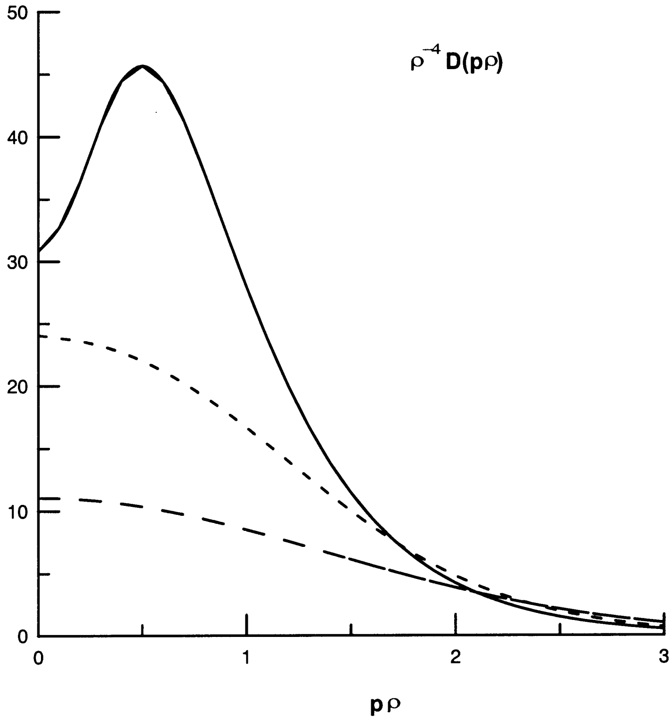

Fig. 6. The invariant functions $\tilde{D}(p)$ as a function of $\rho|p|$ corresponding to the ansatz (44), at different values of the parameter $(\rho\eta_{\rm g})^2$: $(\rho\eta_{\rm g})^2 = 0$ solid line (instanton case), $(\rho\eta_{\rm g})^2 = 0.5$ short dashed line, $(\rho\eta_{\rm g})^2 = 3$ long dashed line

and $\kappa = 0.926$ at $(\eta_{\rm g}\rho)^2 = 3$. In all the cases the parameter κ is close to one in accordance to the fits of lattice data [33]. This means that in a weak background field the dominant role of the $D(x^2)$ function remains as in the SI case; it is close to the SI form at small distances and decays exponentially rapidly at large distances. The $D_1(x^2)$ function is small and positive everywhere; its behavior is very sensitive not only to the external field but also to the gauge phase factor effect[4]. In Appendix A we show that the reason for the smallness of the $D_1(x^2)$ function is an "almost" self-duality property of the CI solutions. Both the invariant functions have two zeros at large distances and at very large x develop positive asymptotics

$$D(x^2), D_1(x^2) \sim |x|^{-3/4} \exp\left(-0.473\,(\eta_{\rm g}\,|x|)^{3/2}\right), \quad (57)$$

where $\eta_{\rm g} \neq 0$ and the constant in the exponent is found numerically.

In order to have contact with the QCD vacuum phenomenology and specify further the instanton-induced model of the gluon correlator, let us discuss the contributions of different terms in (46) to the gluon condensate

$$\begin{aligned}&\left\langle : F_{\mu\nu}[A^{\rm CI}+b]\,(0)\,F_{\mu\nu}[A^{\rm CI}+b]\,(0) :\right\rangle \\ &= \left\langle F^{\rm CI}_{\mu\nu}\,(0)\,F^{\rm CI}_{\mu\nu}\,(0)\right\rangle \\ &\quad + \left\langle : F_{b,\mu\nu}\,(0)\,F_{b,\mu\nu}\,(0) :\right\rangle \\ &\quad + \left\langle : \varDelta F_{\mu\nu}[A^{\rm CI}+b]\,(0)\,\varDelta F_{\mu\nu}[A^{\rm CI}+b]\,(0) :\right\rangle^{\rm interf}.\end{aligned} \quad (58)$$

The background contribution to the gluon condensate

$$\left\langle : F_{b,\mu\nu}\,(0)\,F_{b,\mu\nu}\,(0) :\right\rangle = \left\langle F_b^2\right\rangle_{\rm b} \quad (59)$$

serves as a parameter of the model and, by assumption, is much smaller than the CI contribution given by

$$\left\langle F^{\rm CI}_{\mu\nu}\,(0)\,F^{\rm CI}_{\mu\nu}\,(0)\right\rangle = 32\pi^2 n_{\rm c} N_D^{-1}, \quad (60)$$

where $N_D^{-1} \approx 1$ (see Fig. 4). The interference term after averaging over the relative color orientations and using relations (13) and (17) acquires the form

$$\begin{aligned}&\left\langle : F_{\mu\nu}[A^{\rm CI}+b]F_{\mu\nu}[A^{\rm CI}+b] :\right\rangle^{\rm interf} \\ &= \frac{N_{\rm c}}{16\,(N_{\rm c}^2-1)}\left(32\pi^2 n_{\rm c}\right)\left\langle F_b^2\right\rangle_{\rm b}\int_0^\infty {\rm d}z z^7 \varphi_{\rm g}^2\left(z^2\right)\varPhi(z^2),\end{aligned} \quad (61)$$

where $\varPhi(z^2)$ is defined in (21) (the explicit forms of $\varPhi(z^2)$ are outlined in Appendix B).

The interference term depends on two dimensionless parameters $\alpha_{\rm g} = \rho_{\rm c}\eta_{\rm g}$ and $\beta = \rho_{\rm c}/R$ and the background field condensate can be parameterized as $\left\langle F_b^2\right\rangle_{\rm b}\rho_{\rm c}^4 = \left(9\left(N_{\rm c}^2-1\right)\right)/(a_{\varPhi}N_{\rm c})\alpha_{\rm g}^3\beta$. The instanton size $\rho_{\rm c}$ occurs in the last formula with a high power and leads to indefiniteness of the factor of order 2 in the relation of the external field condensate to the parameters $\alpha_{\rm g}$ and β. To reduce this uncertainty, we can use the physical information about the vacuum properties provided by the QCD SRs and lattice QCD. Indeed, as has been shown in [15], in the SI case there is a relation between the instanton size and the average virtuality of the quarks in the vacuum; see (3). The value of the average quark virtuality has been estimated in a QCD sum rule analysis, $\lambda_q^2 = 0.5\pm0.05\,{\rm GeV}^2$, in [16]; $\lambda_q^2 = 0.4\pm0.1\,{\rm GeV}^2$ in [17], and from the lattice QCD calculations $\lambda_q^2 = 0.55\pm0.05\,{\rm GeV}^2$ in [18]. The relations (3) remain good approximations in the CI case if the external field does not strongly deform the instanton. Numerical calculations of $\lambda_{\rm g}^2$, defined in (3), lead to the estimates

$$\begin{aligned}&\lambda_{\rm g}^2 = 4.8\frac{1}{\rho_{\rm c}^2}(\alpha_{\rm g}^2=0), \quad \lambda_{\rm g}^2 = 5.7\frac{1}{\rho_{\rm c}^2}(\alpha_{\rm g}^2=1), \\ &\lambda_{\rm g}^2 = 7.6\frac{1}{\rho_{\rm c}^2}(\alpha_{\rm g}^2=3).\end{aligned} \quad (62)$$

We show below that the physically motivated background field has a strength parameter $\alpha_{\rm g} \leq 1$ and thus the value of $\lambda_{\rm g}^2$ increases not more than 20%.

The relation for λ_q^2 in (3) can be used to get the scale for the background field condensate

$$\left\langle F_b^2\right\rangle_{\rm b} = \frac{9\left(N_{\rm c}^2-1\right)}{4N_{\rm c}a_{\varPhi}}\left(\lambda_q^2\right)^2\alpha_{\rm g}^3\beta$$

[4] In [34], in the calculations of the invariant functions D, D_1 based on the instanton–anti-instanton ansatz both the influence of the physical vacuum on the instantons and the gluon correlators as the effect of the P exponential factor on the invariant functions has been ignored. As a result, a negative D_1 has been obtained in [34]. However, both facts are important in determining the correct norm and forms of the invariant functions; in particular, in obtaining a small D_1

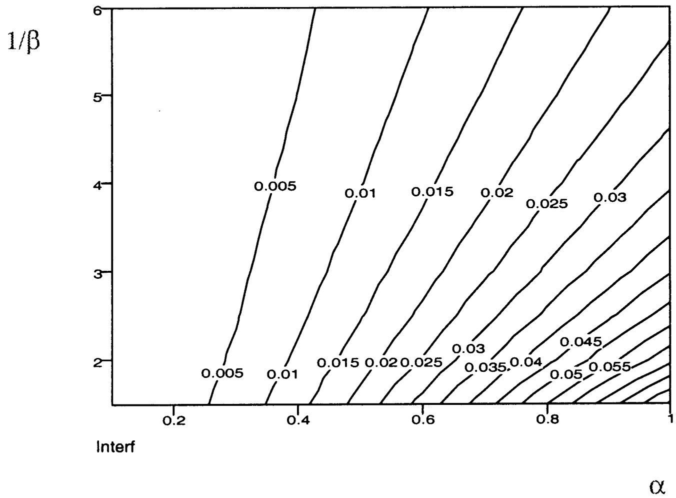

Fig. 7. The interference term contribution to the gluon condensate normalized by the instanton contribution $F_{\mu\nu}[A^I]F_{\rho\sigma}[A^I] = n_c 32\pi^2$ as a function of the large-scale vacuum fluctuation correlation length $1/\beta = R/\rho_c$ and its strength parameter $\alpha_g = \eta_g \rho_c$

and we accept in the following that

$$\lambda_q^2 = 0.5\,\mathrm{GeV}^2.$$

What is the expected range for the parameters α_g and β? By analogy with the instanton liquid vacuum, the parameter β can be interpreted as the ratio of the instanton size to the inter-instanton distance and it is adjusted by $\beta \approx 1/3$. Then the estimate of the upper limit for the strength parameter α_g follows from the assumption that the contribution of the background field to the total gluon condensate $\langle 0 |F^2| 0\rangle_{total} \approx 1\,\mathrm{GeV}^4$ [22] is quite small. This assumption reduces the influence of the model dependent part of correlator and leads to the bound $\alpha_g < 1$, or for the dimensional parameter, $\eta_g^2 = \alpha_g^2 \overline{\lambda}_q^{-2}/4$, $\eta_g < 0.35\,\mathrm{GeV}$. In Fig. 7 we present the values of the interference term as a two parametric plot and see that its contribution to the gluon condensate is small if the short-range and large-scale fluctuations are well separated: $\alpha_g < 1$ and $\beta \ll 1$. For completeness in Appendix B we present the small interference contributions to the invariant forms $A(x^2)$ and $B(x^2)$.

Thus, we construct the model of the gluon correlators. Within this model the invariant functions $A(x^2)$ and $B(x^2)$ are the sum of the short-range instanton-induced contributions (54) and (55) multiplied by the weight factor $32\pi^2 n_c / \langle 0 |F^2| 0\rangle_{total}$ and the long-range contribution (19) modeled by (25) with the weight factor $\langle F_b^2 \rangle_b / \langle 0 |F^2| 0\rangle_{total}$. The parameters of the model are the average instanton size $\rho_c \approx 0.3\,\mathrm{fm}$, the effective instanton density $n_c \approx 1\,\mathrm{fm}^{-4}$, the strength $\langle F_b^2 \rangle_b \le 32\pi^2 n_c$ and the correlation length $R \approx 3\rho_c$ of the background field. The first two parameters are estimated within the instanton liquid models, being in the dilute liquid limit expressed through the vacuum averages: $\rho_c^2 = 2\lambda_q^{-2}$ and $n_c = (2\pi I/N_c)\left(|\langle \overline{q}q\rangle|^2 / \lambda_q^2\right)$, where the numerical constant $I \approx 0.6$ [35]. The latter relation is a consequence of the gap equation [13]. The form of the short-range correlator is defined by ρ_c at small distances and by the long-scale parameter η_g at large distances. The form of the long-range correlator at large distances can be motivated by the results obtained in the dual effective model of QCD [8], where they exponentially decrease (modulo powers) similar to the exponential ansatz in (25). Based on the results of the dual model and lattice measurements one can expect that $A(x^2) \approx B(x^2)$ for the long-range part of the correlator.

The field-strength correlators have been studied on the lattice in [7,10]. On the lattice the following two combinations of invariant functions have been measured:

$$D_\perp\left(x^2\right) = D\left(x^2\right) + D_1\left(x^2\right),$$
$$D_{||}\left(x^2\right) = D\left(x^2\right) + D_1\left(x^2\right) + x^2 \frac{\partial D_1\left(x^2\right)}{\partial x^2},$$

where $D_\perp\left(x^2\right) = 2A\left(x^2\right) - B\left(x^2\right)$ and $D_{||}\left(x^2\right) = B\left(x^2\right)$ in terms of the combinations defined in (53). The lattice measurements of the field-strength correlators are also obtained with the straight line path in the Schwinger exponent. The direct comparison of the model calculations with the lattice data is a delicate problem, since the used parameterization is rather conventional in separating the residual perturbative tail (divergent term $\sim x^{-4}$) from the non-perturbative part (pure exponential finite term). As a result, the fits with ad hoc chosen parameterizations are very unstable with respect to the extraction of the quantities of physical interest: the correlation lengths, the gluon condensate, etc. [33]. The reason is that the perturbative part is strongly divergent (as seen from lattice data), its contribution at small distances would be strongly dependent on the parameterization procedure. On the other hand, by construction we calculate the non-perturbative part of the correlators with perturbative contributions subtracted. In the future, it would be quite desirable to make a new fit to the lattice data using (54) and (55), as an input for the non-perturbative part of the correlators.

At the present stage, we restrict ourselves only to a few qualitative remarks. In [33], the ranges of values of some physical quantities which can be fitted from the lattice data according to different parameterizations were discussed. The correlation length, the gluon condensate and the normalization of invariant functions have been analyzed. As has been noted above, the normalization of invariant functions κ, consistent with small $D_1\left(x^2\right)$, is in agreement with the instanton model and with the fact that the value of the gluon condensate serves as a free model parameter. The values of the gluon condensate extracted from the fits to the lattice data are very sensitive to the parameterization used, being within the interval $\langle : (\alpha_s/\pi) F^a_{\mu\nu} F^a_{\mu\nu} : \rangle = (0.005 - 0.03)\,\mathrm{GeV}^4$. In the lattice "full QCD" fit of the average correlation length l_G of the gluon strength, defined as [33]

$$l_G = \frac{1}{D(0)} \int_0^\infty \mathrm{d}x D\left(x^2\right), \qquad (63)$$

one finds the range $l_G \approx 0.35$–0.45 with lattice quark mass $am_q = 0.01$ and $l_G \approx 0.3$–0.4 fm with $am_q = 0.02$, where a is the lattice unit. One can expect, following a linear extrapolation, that in the chiral limit $am_q \to 0$, $l_G \approx 0.4$–0.5 fm. Now let us omit an important but unsolved problem about the difference of lattice and CI renormalization schemes and norms, to make a rather rough comparison of the corresponding results. The instanton model predicts for the same quantity: $l_G = 0.43$ fm at $\rho_c\eta_g = 0$ (SI), $l_G = 0.37$ fm at $(\rho_c\eta_g)^2 = 1$, $l_G = 0.31$ fm at $(\rho_c\eta_g)^2 = 3$. Thus, the predictions of the instanton model are in qualitative agreement with information extracted from the lattice data, again under the condition $\alpha_g \equiv \rho_c\eta_g < 1$.

4 Conclusions

The instanton model provides a way of constructing the non-local vacuum condensates. We have obtained the expressions for the non-local gluon $\langle : \mathrm{Tr}F^{\mu\nu}(x)\hat{E}(x,y)F^{\rho\sigma}(y)\hat{E}(y,x) : \rangle$ correlator beyond the single instanton (SI) approximation[15]. They have consistent properties both at short and at large distances. The model constructed predicts the behavior of the non-perturbative part of the gluon correlation functions in the short and intermediate region, assuming that it is dominated by the instanton vacuum component.

To this end, we have suggested that the instanton $A_\mu^{\mathrm{CI}}(x)$ is developed in the physical vacuum field $b_\mu(x)$ interpolating large-scale vacuum fluctuations. We have found that at small distances the instanton field dominates, and at large distances it decreases exponentially. We did not assume any particular properties of the long-wave vacuum field $b_\mu(x)$ but managed to reduce the effect to certain phenomenological quantities, namely, the correlation function $\widetilde{B}(x^2)$ determined by its strength $\langle F_b^2 \rangle_\mathrm{b}$ and the correlation length R. Within this model, by averaging over random color vector orientations of the background field with respect to the fixed instanton field orientation, we have found (20) governing the deformation of the instanton under the influence of the weak background vacuum field. Following the idea due to Affleck we have shown that, to stabilize the instanton, we need to put constraints on the system. Next, we have found the constraint-independent asymptotics of the instanton solution at large distances, given by (28) and (29), where it is exponentially suppressed,

$$A_\mu^{CIa}(x) \sim 2\overline{\eta}^a_{\nu\mu}x_\nu \left((\rho\lambda_\mathrm{g})^2\right) / (x^2 |x|^{3/4}) \times \exp\left[-(2/3)\left(\eta_\mathrm{g}|x|\right)^{3/2}\right],$$

unlike the power decreasing SI. It is important to note that the form of this asymptotic behavior is also independent on the model for the background field, and the driven parameter $\eta_\mathrm{g} \sim \left(N_\mathrm{c}/\left(9\left(N_\mathrm{c}^2-1\right)\right)R\langle F_b^2\rangle_\mathrm{b}\right)^{1/3}$ only weakly depends on it. Assuming that the external field is weak, the CI profile function is close to the SI profile at distances smaller than ρ_c and it decreases exponentially at distances larger than η_g^{-1} (see (1)). In particular, this result means that the leading interaction term of a widely separated instanton–anti-instanton pair in the physical vacuum decays exponentially with separation and differs from the dipole interaction term found previously in an unconstrained model. The knowledge of the constraint-independent parts of CI allowed us to construct the solution in the ansatz (31) with the profile functions (42)–(44). As seen from Figs. 2,3 and 4, the profile of the CI and its action are practically independent of the choice of the ansatz if the interference parameter is in the region $\rho_\mathrm{c}\eta_\mathrm{g} < 1$, where our considerations are justified.

Then, for an arbitrary classical gauge field of the form $A_\mu^{\mathrm{CI}a}(x) = 2\overline{\eta}^a_{\nu\mu}x_\nu\varphi_\mathrm{g}\left(x^2\right)$, we have found the expressions (54) and (55) for the combinations of gauge-invariant functions $D\left(x^2\right)$ and $D_1\left(x^2\right)$, which parameterize the gluon field-strength correlator. These expressions generalize the previously known expressions for the SI model [15]. The correlators have been calculated numerically. It turns out, at a reasonable set of parameters, guaranteeing the smallness of the large-scale vacuum field fluctuations, that the $D\left(x^2\right)$ structure is close to the SI induced function with the exponential asymptotics being developed at large distances[5]. At the same time, the $D_1\left(x^2\right)$ structure is about two orders smaller than the $D\left(x^2\right)$ function at any reasonable choice of the parameter $\rho_\mathrm{c}\eta_\mathrm{g}$. As is explained in Appendix A, the reason is that in the dilute vacuum the CIs are “almost” self-dual. The relative strength of the invariant functions $D\left(x^2\right)$ and $D_1\left(x^2\right)$ is very sensitive to the accepted physical picture. The lattice data are in qualitative agreement with predictions of the constrained instanton model. This means, in particular, that in the interpretation of the lattice data a better justified parameterization for the correlation functions can be used. It allows one to extract from the data the values of physical interest and separate the perturbative tail from the non-perturbative contribution. Moreover, due to the fast decay of the CI induced part of the correlators, the exponential decay observed in the lattice calculations can be attributed to the background component of the vacuum field, or can be described by some other field theoretical approaches [8,9]. On the other side, the SI model is inconsistent with the large-distance behavior. The non-perturbative part of the invariant functions $A(x^2)$ and $B(x^2)$ are the sum of the short-range instanton-induced contributions (54) and (55), multiplied by the weight factor $n_\mathrm{c}32\pi^2/\langle 0|F^2|0\rangle_\mathrm{total}$, and the long-range contribution (19), modelled by the exponentially decreasing function (25) with the weight factor $\langle F_b^2\rangle_\mathrm{b}/\langle 0|F^2|0\rangle_\mathrm{total}$.

The constrained instanton model introduces two characteristic scales (correlation lengths). One is related to the short distance behavior of the correlation functions and the other with the long-range distance behavior. The first one, λ_g^{-1}, is predictable and is expressed in terms of physical quantities. In the SI approximation, given by (3) and (4), it is proportional to the instanton size, and

[5] A similar behavior is expected for the quark correlator in the physical vacuum[36].

gains small negative corrections due to the background; see (62). As to the large scale it is out of the scope of our model, and we can only physically relate it to the confinement size or extract it from the long-distance asymptotics of lattice calculations. The microscopic description of the long-distance background field needs other considerations not examined in the present work.

The calculations have been performed in a gauge-invariant manner by using the expressions for the instanton field in the axial gauge. The behavior of the correlation functions demonstrates that in the single constrained instanton approximation the model of non-local condensates can well reproduce the behavior of the functions at short and intermediate distances, while the large-scale asymptotics is dominated by the background field. It would be quite desirable to make a fit to the lattice data using as an input the instanton-induced correlators. The important question concerning the interacting ensemble of the constrained instantons also has to be postponed for other specific work.

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Appendix A

The question may arise why the short-range vacuum contributions to the $D_1\left(x^2\right)$ structure are negligible. As was found in [15], in the SI case $D_1\left(x^2\right)=0$ due to the self-duality of the instanton solution. The CIs are not self-dual and contribute to $D_1\left(x^2\right)$, but to what extent is the self-duality violated? We are going to show that for the reasonable set of parameters and ansätze assumed the CIs are "almost" self-dual and this is the reason for the smallness of $D_1\left(x^2\right)$. On the contrary, if lattice simulations detected a very big contribution to $D_1\left(x^2\right)$, this would mean that self-duality is lost and there is no chance to save instantons as individual objects in the QCD vacuum. From the point of view of our model, any essential contributions to $D_1\left(x^2\right)$ can arise only from the large-scale vacuum fluctuations.

The dual field strength $\widetilde{F}_{\mu\nu}^{\mathrm{CI},a}(x)=(1/2)\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}^{\mathrm{CI},a}(x)$, where $F_{\mu\nu}^{\mathrm{CI},a}(x)$ is defined in (33), can be expressed in the form

$$\widetilde{F}_{\mu\nu}^{\mathrm{CI},a}(x)=4\left[\overline{\eta}_{\mu\nu}^a\widetilde{\omega}_1(x)+\left(x_\mu\overline{\eta}_{\nu\rho}^a-x_\nu\overline{\eta}_{\mu\rho}^a\right)x_\rho\widetilde{\omega}_2(x)\right], \tag{A.1}$$

where

$$\widetilde{\omega}_1(x)=\varphi_{\mathrm{g}}\left(x^2\right)+x^2\frac{\partial\varphi_{\mathrm{g}}\left(x^2\right)}{\partial x^2},$$
$$\widetilde{\omega}_2(x)=\omega_2(x)=\varphi_{\mathrm{g}}^2\left(x^2\right)+\frac{\partial\varphi_{\mathrm{g}}\left(x^2\right)}{\partial x^2}. \tag{A.2}$$

Let us consider the difference of the field strengths given in the regular gauge:

$$F_{\mu\nu}^{\mathrm{CI}a}(x)-\widetilde{F}_{\mu\nu}^{\mathrm{CI}a}(x)=4\left[\overline{\eta}_{\mu\nu}^a+2\frac{\left(x_\mu\overline{\eta}_{\nu\rho}^a-x_\nu\overline{\eta}_{\mu\rho}^a\right)x_\rho}{x^2}\right]\omega_2^{\mathrm{reg}}(x)\,x^2. \tag{A.3}$$

The self-duality condition $F_{\mu\nu}^{\mathrm{CI}a}(x)-\widetilde{F}_{\mu\nu}^{\mathrm{CI}a}(x)=0$ is satisfied for the SI case, where $x^2\omega_2^{\mathrm{reg},I}(x)=0$ (see (40)). Comparing (33) and (A.1) with (A.3) we can consider the condition

$$\left|\omega_2^{\mathrm{reg,CI}}(x)\right|x^2<<\left|\omega_1^{\mathrm{reg,CI}}(x)\right|\sim\left|\omega_1^{\mathrm{reg},I}(x)\right|$$

as a criterion indicating that the field is "almost" self-dual. It can be checked numerically that this is really the case at a reasonable choice of the background field-strength parameter $\alpha_{\mathrm{g}}<1$ and all forms of ansätze for CI. At the same time at larger values of the parameter $\alpha_{\mathrm{g}}\simeq 4\div 6$ the inequality is not fulfilled. Thus we show that assuming diluteness of the vacuum the CIs are "almost" self-dual solutions and, as a consequence, contribute very little to the $D_1\left(x^2\right)$ structure of the vacuum gluon field-strength correlators.

Appendix B

The function $\Phi(z^2)$ for the different forms of the invariant function $\widetilde{B}\left(x^2\right)$:

$$\Phi_G(z^2)=\frac{2}{3y^4}\left[2\sqrt{\pi}y^3\mathrm{erf}(y)-3y^2+1-\left(1-2y^2\right)\exp\left(-y^2\right)\right],\quad\text{Gaussian}, \tag{B.1}$$

$$\Phi_M(z^2)=\frac{2}{3y^4}\left[4y^3\arctan(y)+y^2-\left(1+3y^2\right)\ln\left(1+y^2\right)\right],\quad\text{monopole}, \tag{B.2}$$

and

$$\Phi_E(z^2)=\frac{4}{3y^4}\left[2y^3-3y^2+6-6(1+y)\exp(-y)\right],\quad\text{exponential}, \tag{B.3}$$

where $y=z/R$, R being the correlation length of the large-scale vacuum field.

For completeness we present here the small interference contribution to the invariant forms $A(x^2)$ and $B(x^2)$

$A^{\mathrm{interf}}(x^2)$

$$= N_D \left\langle 0 \left| F_b^2 \right| 0 \right\rangle \frac{N_c}{6\left(N_c^2-1\right)} \int_0^\infty \mathrm{d}rr^2 \times \int_0^\infty \mathrm{d}t\Phi(z_+, z_-)\varphi_g\left(z_+^2\right)\varphi_g\left(z_-^2\right) \cdot (z_+\cdot z_-)\left\{(z_+\cdot z_-)\left(3-4\sin^2(\alpha_z)\right) - xr\sin(2\alpha_z)\right\}, \quad \text{(B.4)}$$

$$B^{\text{interf}}(x^2) = N_D \left\langle 0 \left| F_b^2 \right| 0 \right\rangle \frac{N_c}{6\left(N_c^2-1\right)} \times \int_0^\infty \mathrm{d}rr^2 \int_0^\infty \mathrm{d}t\Phi(z_+, z_-)\varphi_g\left(z_+^2\right)\varphi_g\left(z_-^2\right) \cdot r^2\left\{2(z_+\cdot z_-)\left(2-3\sin^2(\alpha_z)\right) - xr\sin(2\alpha_z)\right\},$$

where

$$\Phi(z_+, z_-) = 4\int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta\alpha\beta\widetilde{B}\left[(\alpha z_+ - \beta z_-)^2\right],$$

$z_\pm = (r, t\pm(x/2))$, $z = (r,t)$, and $\varphi_g\left(z^2\right)$ is defined in (31). In deriving the above expressions we used the Schwinger–Fock gauge for the background field, (16)–(19), and neglected the derivative $\widetilde{D}_1'(x^2)$ in comparison with the invariant function $\widetilde{B}(x^2)$ itself. In fact, this is consistent with $\widetilde{D}_1\left(x^2\right) = 0$ which follows from the lattice data [7] and instanton calculations [15]. We find that the interference contributions to the correlators are very small in absolute value, have a shorter correlation length compared with the CI contributions (54) and (55), and do not lead to an obvious appearance of the $D_1\left(x^2\right)$ structure.

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